Generating Gomory's Cuts for linear integer programming problems:
the HOW and WHY

A Gomory's Cut is a linear constraint with the property that it is strictly stronger than its Parent, but it does not exclude any feasible integer solution of the LP problem under consideration. It is used, in conjunction with the Simplex Method, to generate optimal solutions to linear integer programming problems (LIP). Formally the LP and LIP problems under consideration are as follows:

LP: opt $c^t x$ subject to $Ax = b$, $x >= 0$

LIP: opt $c^t x$ subject to $Ax = b$, $x >= 0$ and integer

We refer to LP as the linear programming relaxation of LIP. Needless to say, if the optimal solution to LP satisfies the integrality constraint of LIP then it must also be optimal with respect to LIP. In such a case we do not need any cuts, in fact we can completely relax and have a good cup of coffee.

We are interested in Gomory's Cuts in cases where the optimal solution we have for LP does not satisfy the integrality constraint of ILP. That is, we are dealing here with a situation where the optimal solution to LP is such that in the final Simplex Tableau of the LP problem we have a row whose RHS value is not an integer. This row represents a linear equality constraint whose RHS value is not an integer.

As an example, consider the minute ILP problem

$$z^*: = \max z = 5x_1 + 8x_2$$
s.t.
$$x_1 + x_2 <= 6$$
$$5x_1 + 9x_2 <= 45$$
$$x_1, x_2 >=0, \text{ and integer}$$

Observe that all the coefficient of the constraints - including the RHS values - are inegers.

In the usual LP manner we rewrite this problem as follows:

$$z^*: = \max z = 5x_1 + 8x_2$$
s.t.
$$x_1 + x_2 + s_1 = 6$$
$$5x_1 + 9x_2 + s_2 = 45$$
$$x_1, x_2, s_1, s_2 >=0, \text{ and integer}$$

where s_1 and s_2 are the slack variables associated with the two functional constraints. Note that the integrality also applies to these variables.

The optimal solution to the linear programming relaxation of this problem is x = (2.25,3.75). The final Simplex Tableau for this problem is as follows:

BV	x1	x2	s1	s2	RHS
x1	1	0	2.25	-0.25	2.25
x2	0	1	-1.25	0.25	3.75
z	0	0	1.25	0.75	41.25

The situation is described graphically in Figure 1.

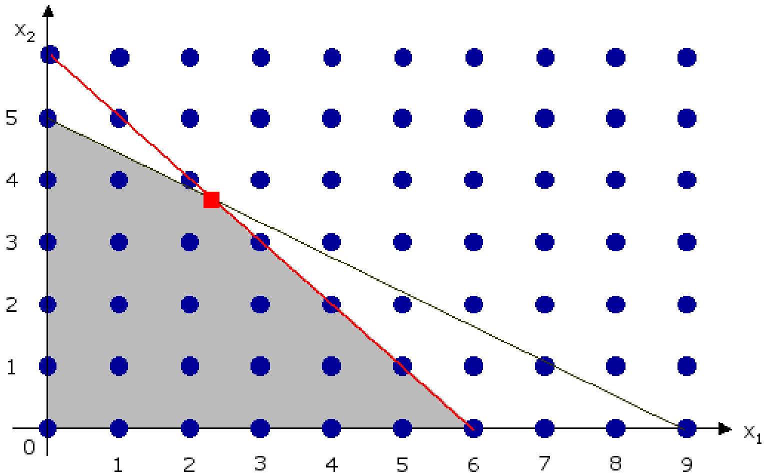

Figure 1

The second row in the final Simplex Tableau represents the following constraint:

$$0x_1 + x_2 - 1.25s_1 + 0.25s_2 = 3.75$$

This constraint generates the following Gomory's Cut:

$$0.75s_1 + 0.25s_2 >= 0.75$$

Don't panic if you do not understand where this constraint came from. We discuss the construction of such constraints downstairs. The point to note at the moment is that if we add this constraint to the LIP original problem, we obtain the following new ILP problem:

$$z^*: = \max z = 5x_1 + 8x_2$$

s.t.

$$x_1 + x_2 + s_1 = 6$$
$$5x_1 + 9x_2 + s_2 = 45$$
$$0x_1 + 0x_2 + 0.75s_1 + 0.25s_2 >= 0.75$$
$$x_1, x_2, s_1, s_2 >=0, \text{ and integer}$$

This is illustrated in Figure 2. The broken blue line represents the hyperplane induced by this new constraint (in the x plane).

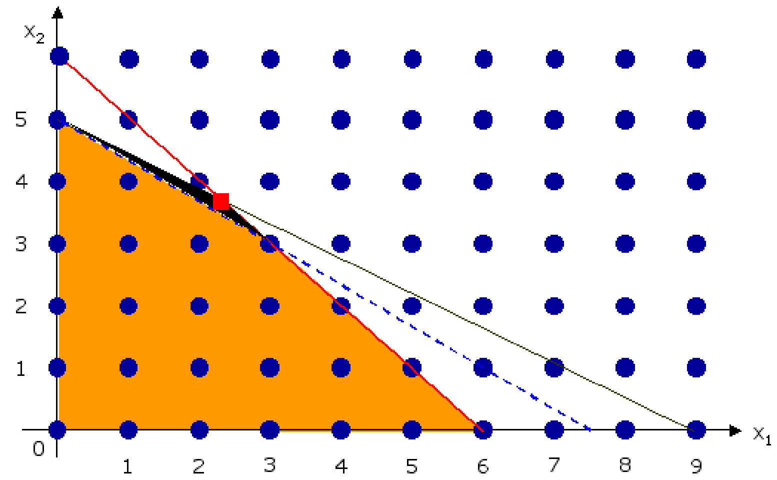

Figure 2

The new (reduced) feasible region has two important properties:

- It does not contain the the optimal basic feasible solution to the old LP problem obtained by the Simplex Method. That is, the cut removed the optimal solution to the LP problem.
- It contains all the feasible solutions to the original ILP problem. That is, the cut did not remove any feasible solution of the original integer programming problem.

This is not an accident. Indeed, these two properties characterize Gomory's Cuts.

> It should be pointed out that the objective of this discussion is not to provide you, Dear Reader, with a tutorial on Gomory's Cutting Plane Method. Rather, the objective of this discussion is to focus on the cuts themselves, in particular on how they are generated. Consult your favourite OR textbook for details on how the cuts are used to solve ILP problems. For the purposes of this discussion suffice it to indicate that the Method repeatedly adds cuts to the original LP problem until the optimal solution satisfies the integrality constraint.

As illustrated by the above example, to generate a Gomory's Cut we need a linear equality constraint with the following property: the RHS value of the constraint is not an integer. We represent such as generic constraint as follows:

$$w_1x_1 + \ldots + w_nx_n = I + f$$

where I is an (non-negative) integer and f is a positive fraction. For example,

$$3\ x_1 + 3\ 1/4\ x_2 + 5\ 7/8\ x_2 = 6\ 2/3$$

is such a case where I = 6 and f =2/3. Note that because all the decision variables must be integers, if the RHS of the constraint is not an integer, then at least one of the LHS coefficients ($w_1,\ldots,w_n$) is not an integer.

Note:
As indicated above, we shall use proper fractions throughout the exposition. Our convention for displaying fractions is to use a single space to separate the integer part and the fractional part of a number. Thus, we write 2 2/3 to represent the number whose value is 2 + (2/3) and we write -2 2/3 to represent the number whose value is - (2 + (2/3)). Hence, according to this convention 2 1/4 is equal to 2.25 and -2 1/4 is equal to -2.25. Make sure you fully understand this convention (no big deal!) before you proceed.

There are two related yet distinct issues here, namely: How do we generate Gomory's Cuts? And Why do we do it in this particular way? In this short discussion we briefly address these issues.

The HOW.

Consider the linear equality constraint

$$3\ x_1 + 1\ 1/4\ x_2 + 1/3\ x_3 = 2\ 2/3$$

where the decision variables x_1, x_2, x_3 are non-negative integers.

We rewrite this constraint by separating each of the coefficient into two parts: an integer part and a fraction. The fraction is required to be non-negative. If there is no fractional part, we use 0/1 as a dummy fraction.

Thus, according to this scheme we re-write the above constraint as follows:

$$(3 + 0/1)x_1 + (1 + 1/4)x_2 + (0 + 1/3)x_3 = 2 + 2/3$$

Next, we collect all the integer parts (including the integer part of the RHS) on the left hand side of the equation. So our constraint is re-written thus:

$$0/1\ x_1 + 1/4\ x_2 + 1/3\ x_3 + (3\ x_1 + x_2 + 0\ x_3 - 2) = 2/3$$

Now we drop all the integers terms on the left hand side of the equation and change the equation into a >=. For our beloved constraint this yields:

$$0/1\ x_1 + 1/4\ x_2 + 1/3\ x_3 + \sout{(3\ x_1 + x_2 + 0\ x_3 - 2)} >= 2/3$$

namely

$$0/1\ x_1 + 1/4\ x_2 + 1/3\ x_3 >= 2/3$$

This is the Gomory's Cut induced by our original constraint.

Before we formulate recipes for this process, let us consider the following constraint, observing that one of the coefficients is negative.

$$3\ x_1 - 1\ 1/4\ x_2 + 1/3\ x_3 = 8/3$$

As above, we first separate the coefficients into two parts (integers and fractions), recalling that the fractions are required to be non-negative:

$$(3 + 0/1)x_1 + (-2 + 3/4)x_2 + (0 + 1/3)x_3 = 2 + 2/3$$

Next, we collect all the integers on the left hand side:

$$0/1\ x_1 + 3/4\ x_2 + 1/3\ x_3 + (3\ x_1 - 2\ x_2 + 0\ x_3 - 2) = 2/3$$

Finally we drop the integer terms on the right hand side and change the equation into a >=. Here is the final product for our constraint:

$$0/1\ x_1 + 3/4\ x_2 + 1/3\ x_3 >= 2/3$$

So here is the recipe for the process of constructing a Gomory's Cut for a give linear constraint:

Gomory's Cut Recipe: Long Version

- Step 1: Express each of the coefficients of the constraint as the sum of an integer and a non-negative fraction. Note that this representation is unique.

- Step 2: Collect all the integer terms on the left hand side of the equation (do not forget to change the sign!)
- Step 3: Drop all the integer terms and change the = to >=.

As far as the final result is concerned Step 2 is just a nuisance. It is introduced above only to explain the logic behind the process. If we drop this step, we obtain the following:

Gomory's Cut Recipe: Short Version

- Step 1: Express each of the coefficients of the constraint as the sum of an integer and a non-negative fraction. Note that this representation is unique.
- Step 2: Drop all the integer terms and change the = to >=.

The following example illustrates the Short version of the recipe in action.

Example

Consider the linear equality constraint

$$0\ x_1 - 3/4\ x_2 + 1/3\ x_3 - 3\ 1/4\ x_4 = 10\ 3/7$$

Step 1:

$$(0 + 0/1)x_1 + (-1 + 1/4)x_2 + (0 + 1/3)\ x_3 + (-4 + 3/4)x_4 = 10\ 3/7$$

Step 2:

$$0/1\ x_1 + 1/4\ x_2 + 1/3\ x_3 + 3/4\ x_4 >= 3/7$$

This is the Gomory's Cut of our constraint. Note that , as expected, all the coefficients - including the RHS value - of the cut are non-negative fractions (zeros are welcome).

As they say, Life is beautiful!

There is yet another version of the recipe to consider. This version focuses on the result rather than the process. Here it is:

Gomory's Cut Recipe: the Bottom Line

1. Change each of the coefficients of the linear constraints - including the RHS - as follows:

Rule	Original constraint	Gomory's Cut
Rule 1	Integer (positive or negative)	Zero
Rule 2	Positive fraction eg. 1/3	Unchanged 1/3
Rule 3	Negative fraction eg. -1/3	Positive complement 2/3

2. Change the = to >=.

Examples

Original constraint	Gomory's Cut
$3 x_1 + 3\ 2/5\ x_2 - 2/5\ x_3 = 8\ 3/4$	$2/5\ x_2 + 3/5\ x_3 >= 3/4$
$-3\ 1/4\ x_1 + 2/5\ x_2 - 2/5\ x_3 = 7\ 5/6$	$3/4\ x_1 + 2/5\ x_2 + 3/5\ x_3 >= 5/6$

The following module generates Gomory's Cuts in accordance with the short recipe given above.

Gomory's Cut for Dummies

- Enter raw w_j values in the appropriate cells, then press the Step 1 button.
- You are welcome to use the pre-set examples.
- Click the radio buttons for explanations on how the coefficients of the cut are computed.

Examples reset

	x_1	x_2	x_3	x_4	x_5	x_6		RHS
w_j							=	
Step 1							=	
Step 2							>=	
Explain	○	○	○	○	○	○		○

The WHY?

A Gomory's Cut has the following two important properties:

- It cuts the feasible region of the LP problem hence makes it smaller.
- It does not affect the set of feasible solutions to the integer programming problem.

These observations are direct implications of the following obvious result:

Theorem

Let IL and IR be any two integers, f a strictly positive fraction and F the sum of strictly positive fractions such that

$$IL + F = IR + f$$

Then

$$IL <= IR$$
$$F >= f$$

Proof. Let (IL,IR,f,F) be any quadruplet satisfying the conditions stipulated above. Since IL and IR are integers, F is the sum of strictly positive fractions and f is a strictly positive fraction, we must have F >= f. In fact, it is clear that if F < 1 then F = f and if F > 1 then F >f. Note that because f is a fraction and IL and IR are integers, F cannot be an integer. In particular F cannot be equal to 1. Thus, in view of IL + F = IR + f we conclude that IL <= IR. In fact, it is obvious that F < 1 implies IL = IR and F > 1 implies IL < IR. Q.E.D

Exercise:

Show that Gomory's Cuts possess the two properties mentioned above.

Remarks:

- Gomory described his method in a very short paper entitled "Outline of an algorithm for integer solutions to linear programs", *Bulletin of the American Mathematical Society*, 64, pp. 275-278, 1958.
- Some details on Gomory's professional career (and a recent photo) can be found here.